

On some algebraic examples of Frobenius manifolds ^{*}

A.E. Mironov [†] I.A. Taimanov [‡]

1 Introduction

In this paper we demonstrate how to construct explicit examples of Frobenius manifolds by using analytical methods of finite-gap integration. There-with we apply Krichever’s scheme of constructing solutions to the associativity equations [5]. Although it is rather clear that the solutions to associativity equations corresponding to smooth spectral curves are not quasihomogeneous we show that in a very degenerate case when the spectral curve consists of rational irreducible components one may construct quasihomogeneous solutions to these equations. The extension of these solutions to Frobenius manifolds is achieved by using some technical algebraic lemma which is exposed in §5.

Until recently all known Frobenius manifolds were given by original Dubrovin’s examples of Frobenius structures on the spaces of orbits of the Coxeter groups (in this case Dubrovin used the Saito flat metric on the space of orbits and such solutions to the WDVV equations corresponding to the A_n singularities were found in [2]) and on the Hurwitz spaces, by quantum cohomology, and by the extended moduli space of complex structures on Calabi–Yau manifolds [1]. In [8] this list was expanded by Shramchenko who “doubled” Frobenius structures by Dubrovin on the Hurwitz spaces (Shramchenko’s manifolds have twice the dimension of the Hurwitz spaces).

In all these cases the manifold with such a structure has its own specified geometrical meaning and only quantum cohomology can be not semisimple, i.e. contain nilpotent elements in a tangent Frobenius algebra at a generic

^{*}The work was supported by RFBR (no. 06-01-00094a) and the complex integration project 2.15 of SB RAS. The first author (A.E.M.) was also supported by the grant of President of Russian Federation (grant MK-9651.2006.1) and Russian Science Support Foundation.

[†]Institute of Mathematics, 630090 Novosibirsk, Russia; mironov@math.nsc.ru

[‡]Institute of Mathematics, 630090 Novosibirsk, Russia; taimanov@math.nsc.ru

point. Our examples always lack the semisimplicity property (thus they are not directly related to isomonodromic deformations, see [4]) and are obtained by analytical methods without any recognition of their relations to other geometrical objects. These examples are algebraic in the sense that the correlators $c_{ijk} = \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k}$ are algebraic functions.

2 Some preliminary facts on Egoroff metrics and Frobenius manifolds

Given a symmetric tensor $\eta^{\alpha\beta} = \eta^{\beta\alpha}$, the associativity equations for the function F take the form

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\delta \partial t^\mu} = \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\delta \partial t^\mu}, \quad (1)$$

where $t = (t^1, \dots, t^n)$ and the indices range from 1 to n . They are equivalent to the condition that the finite-dimensional algebra with generators e_1, \dots, e_n and the commutative multiplication

$$e_\alpha \cdot e_\beta = c_{\alpha\beta}^\gamma e_\gamma, \quad c_{\alpha\beta\gamma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \quad c_{\alpha\beta}^\gamma = \eta^{\gamma\delta} c_{\alpha\beta\delta},$$

is associative with respect to the multiplication, i.e. we have

$$(e_\alpha \cdot e_\beta) \cdot e_\gamma = e_\alpha \cdot (e_\beta \cdot e_\gamma) \quad \text{for all } \alpha, \beta, \gamma.$$

These equations first appeared in the topological field theory where together with conditions

$$c_{1\alpha\beta} = \eta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, n; \quad \eta^{\alpha\beta} \eta_{\beta\gamma} = \delta_\gamma^\alpha,$$

with $\eta_{\alpha\beta}$ a constant metric, probably indefinite, and

$$F(\lambda^{d_1} t^1, \dots, \lambda^{d_n} t^n) = \lambda^{d_F} F(t^1, \dots, t^n) \quad (2)$$

(the quasihomogeneity condition) they the system of Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations [9, 2].

The quasihomogeneity condition is generalized as follows: it is assumed that there is the vector field $E = (q_\beta^\alpha t^\beta + r^\alpha) \partial_\alpha$ such that $E^\alpha \partial_\alpha F = d_F F$ (in the case of (2) we have $E = d_1 t^1 \partial_1 + \dots + d_n t^n \partial_n$) and this generalization covers the case of quantum cohomology.

Since, by [2], it is only important for the correlators c_{ijk} , i.e. third derivatives of F , to be quasihomogeneous in the sense of (2) there is another generalization of quasihomogeneity which reads that

$$E^\alpha \partial_\alpha F = d_F F + (\text{a polynomial of second order in } t^1, \dots, t^n).$$

This generalization is important for us because in our examples part of exponents d_i equal to -1 .

The geometric counterpart of a solution to the WDVV equations is a Frobenius manifold which notion was introduced by Dubrovin [4] who discovered rich differential-geometrical properties of the WDVV equation and thus gave rise to the Frobenius geometry.

There is an important relation between Frobenius manifolds and Egoroff metrics also discovered by Dubrovin [3].

A metric

$$ds^2 = \sum_{i=1}^n H_i^2(u) (du^i)^2 \quad (3)$$

is called Egoroff if the rotation coefficients $\beta_{ij} = \frac{\partial_i H_j}{H_i}$, $i \neq j$, are symmetric: $\beta_{ij} = \beta_{ji}$. Let us consider the Darboux–Egoroff metrics, i.e., flat Egoroff metrics

$$\eta_{\alpha\beta} dx^\alpha dx^\beta = \sum_{i=1}^n H_i^2(u) (du^i)^2$$

where x^1, \dots, x^n are flat coordinates in some domain where the coefficients $\eta_{\alpha\beta}$ are constant. We have $\eta^{\alpha\beta} = \sum_i H_i^{-2} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^i}$ and the flatness condition together with symmetry of the rotation coefficients imply that there is a function F called the prepotential such that

$$c_{\alpha\beta\gamma} = \sum_i H_i^2 \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta} \frac{\partial u^i}{\partial x^\gamma} = \frac{\partial^3 F}{\partial x^\alpha \partial x^\beta \partial x^\gamma} \quad (4)$$

and the associativity equations hold:

$$c_{\alpha\beta}^\lambda c_{\lambda\gamma}^\mu = c_{\alpha\lambda}^\mu c_{\beta\gamma}^\lambda \quad \text{for all } \alpha, \beta, \gamma = 1, \dots, n,$$

where

$$c_{\beta\gamma}^\alpha = \sum_i \frac{\partial x^\alpha}{\partial u^i} \frac{\partial u^i}{\partial x^\beta} \frac{\partial u^i}{\partial x^\gamma}.$$

The inverse is also true assuming that this associative algebra is semisimple: one may construct from such a solution $F(t)$ to the associativity equations a Egoroff metric meeting (4).

3 Finite gap construction of Egoroff metrics and Frobenius manifolds

The condition that the formula (3) defines the Euclidean metric $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = \delta_{\alpha\beta} dx^\alpha dx^\beta$ in some domain (without assuming that the rotation coefficients are symmetric) means that u^1, \dots, u^n are curvilinear n -orthogonal coordinates in this domain and it is written in the form of the Darboux equations.

First the methods of integrable systems were applied for constructing explicit solution to the Darboux system by Zakharov [10] who used the dressing method and then this approach was extended by Krichever onto the finite gap integration method [5].

In [7] we already applied Krichever's procedure to a very degenerate case when the spectral curve is reducible and all its reducible components are rational. In this case the procedure of constructing solutions reduces to linear equations.

We consider the same spectral curves in this paper.

Let Γ be a reducible algebraic curve such that every of its irreducible components $\Gamma_1, \dots, \Gamma_s$ is isomorphic to \mathbb{CP}^1 and all singularities on Γ are intersections of different components.

A regular differential Ω on Γ is defined by meromorphic differentials $\Omega_1, \dots, \Omega_s$ on the components such that every such a differential may have poles only simple poles and only at the intersection points of the components and the sum of the residues at every intersection point vanishes: $\sum_{j=1}^r \text{resp} \Omega_{i_j} = 0, \quad P \in \Gamma_{i_1} \cap \dots \cap \Gamma_{i_r}.$

Let us take three divisors on Γ :

$$P = P_1 + \dots + P_n, \quad D = \gamma_1 + \dots + \gamma_{g_a+l-1}, \quad R = R_1 + \dots + R_l,$$

where g_a is the arithmetic genus of Γ . Let us denote by k_i^{-1} some local parameter near P_i , $i = 1, \dots, n$. It is said that $\psi(u^1, \dots, u^n, z)$, $z \in \Gamma$, is the Baker–Akhiezer function corresponding to the data $S = \{P, D, R\}$ if

- 1) $\psi \exp(-u^i k_i)$ is analytic near P_i , $i = 1, \dots, n$;
- 2) ψ is meromorphic on $\Gamma \setminus \{\cup P_i\}$ with poles at γ_j , $j = 1, \dots, g_a + l - 1$;
- 3) $\psi(u, R_k) = 1$, $k = 1, \dots, l$.

Let us take an additional divisor $Q = Q_1 + \dots + Q_n$ on Γ such that $Q_i \in \Gamma \setminus \{P \cup D \cup R\}$, $i = 1, \dots, n$ and put

$$x^j(u^1, \dots, u^n) = \psi(u^1, \dots, u^n, Q_j), \quad j = 1, \dots, n.$$

For such curves the Krichever scheme works as follows [7]:

- Let Γ admit a holomorphic involution $\sigma : \Gamma \rightarrow \Gamma$ such that
 - 1) σ has exactly $2m, m \leq n$, fixed points which are just $P_1, \dots, P_n \in P$ and $2m - n$ points from Q ;
 - 2) $\sigma(Q) = Q$, i.e. the non-fixed points from Q are interchanged by the involution:

$$\sigma(Q_k) = Q_{\sigma(k)}, \quad k = 1, \dots, n;$$

- 3) $\sigma(k_i^{-1}) = -k_i^{-1}$ near $P_i, i = 1, \dots, n$;
- 4) there exists a regular differential Ω on Γ such that its divisors of zeros and poles have the form

$$(\Omega)_0 = D + \sigma D + P, \quad (\Omega)_\infty = R + \sigma R + Q.$$

Then Ω is a pullback of some meromorphic differential Ω_0 on $\Gamma_0 = \Gamma/\sigma$ and we have

$$\sum_{k,l} \eta_{kl} \partial_{u^i} x^k \partial_{u^j} x^l = \varepsilon_i^2 h_i^2 \delta_{ij},$$

where

$$h_i = \lim_{P \rightarrow P_i} \left(\psi e^{-u^i k_i} \right), \quad \eta_{kl} = \delta_{k, \sigma(l)} \text{res}_{Q_k} \Omega_0,$$

and

$$\Omega_0 = \frac{1}{2} \left(\varepsilon_i^2 \lambda_i + O(\lambda_i) \right) d\lambda_i, \quad \lambda_i = k_i^{-2}, \quad \text{at } P_i, i = 1, \dots, n.$$

Moreover if there is an antiholomorphic involution $\tau : \Gamma \rightarrow \Gamma$ such that all fixed points of σ are fixed by τ and

$$\tau^*(\Omega) = \overline{\Omega}$$

then the coefficients $H_i(u)$ are real valued for $u^1, \dots, u^n \in \mathbb{R}$ and u^1, \dots, u^n are n -orthogonal coordinates in the flat n -space with the metric $\eta_{kl} dx^k dx^l$.

The proof of this statement is basically the same as Krichever's original proof for the case of smooth spectral curves [5]. It is only necessary to consider regular differentials instead of meromorphic and specialize for g the arithmetic genus which is different from the geometric genus for singular curves.

The following theorem distinguishes some special case when this construction leads to Darboux–Egoroff metrics and quasihomogeneous solutions to the associativity equations.

Theorem 1 1) Let every component $\Gamma_i, i = 1, \dots, n$, contain a pair of points $P_i = \infty, Q_i = 0$ and $k_i^{-1} = z_i$ be a global parameter on Γ_i . Let us also assume that any intersection point $a \in \Gamma_i \cap \Gamma_j$ of different components has the same coordinates on both components:

$$z_i(a) = z_j(a)$$

and the involution σ takes the form

$$\sigma(z_i) = -z_i.$$

Then the metric

$$ds^2 = \eta_{kl} dx^k dx^l = \sum_i (\varepsilon_i^2 h_i^2) (du^i)^2, \quad h_i = h_i(u^1, \dots, u^n), \quad i = 1, \dots, n,$$

constructed from these spectral data is a Darboux–Egoroff metric.

2) Moreover assume that the spectral curve is connected and the Baker–Akhiezer function is normalized just at one point r :

$$\psi(u, r) = 1, \quad R = r \in \Gamma.$$

Then the functions

$$c_{\alpha\beta\gamma}(x) = \sum_{i=1}^n H_i^2 \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta} \frac{\partial u^i}{\partial x^\gamma}, \quad H_i = \varepsilon_i h_i,$$

are homogeneous

$$c_{\alpha\beta\gamma}(\lambda x^1, \dots, \lambda x^n) = \frac{1}{\lambda} c_{\alpha\beta\gamma}(x^1, \dots, x^n).$$

Proof of the first statement follows Krichever’s scheme [5]. We take the meromorphic function $f : \Gamma \rightarrow \mathbb{C}$ defined by the parameters $z_i, i = 1, \dots, n$, on the components:

$$f(w) = z_i(w) \quad \text{for } w \in \Gamma_i.$$

Then the differential

$$\omega = f(z) \frac{\partial_i \psi(u, z)}{h_i(u)} \frac{\partial_j \psi(u, \sigma(z))}{h_j(u)}$$

has poles only at P_i and P_j with the residues β_{ij} and $-\beta_{ji}$ which implies

$$\sum \text{res } \omega = \beta_{ij} - \beta_{ji} = 0.$$

Proof of the second statement immediately follows from Lemmata 1 and 2.

Lemma 1 *Under the assumptions of Theorem 1, we have the equality*

$$x^j(u^1 + \mu, \dots, u^n + \mu) = e^{-r\mu} x^j(u^1, \dots, u^n).$$

Proof. On the component Γ_j the function equals

$$\psi_j(z_j) = e^{u^j z_j} \left(f_{j0}(u) + \frac{f_{j1}(u)}{z_j - \gamma_1^j} + \dots + \frac{f_{jk_j}(u)}{z_j - \gamma_{k_j}^j} \right).$$

Let $r \in \Gamma_p$. Then the condition $\psi(r) = 1$ is written as

$$f_{p0}(u) + \frac{f_{p1}(u)}{r - \gamma_1^p} + \dots + \frac{f_{pk_p}(u)}{r - \gamma_{k_p}^p} = e^{-ru^p}. \quad (5)$$

If the components Γ_i and Γ_j intersect at some point a then this point has the same coordinates on both components and the condition

$$\psi_j(a) = \psi_i(a),$$

takes the form

$$\begin{aligned} e^{a(u^j - u^i)} \left(f_{j0}(u) + \frac{f_{j1}(u)}{a - \gamma_1^j} + \dots + \frac{f_{jk_j}(u)}{a - \gamma_{k_j}^j} \right) = \\ \left(f_{i0}(u) + \frac{f_{i1}(u)}{a - \alpha_1^i} + \dots + \frac{f_{ik_i}(u)}{a - \alpha_{k_i}^i} \right). \end{aligned} \quad (6)$$

By (5) and (6), the translation

$$u^j \rightarrow u^j + \mu,$$

results in the multiplication of the coefficients f_{sk} :

$$f_{sk} \rightarrow f_{sk} e^{-r\mu} \quad \text{for all } s, k.$$

Since $x^j(u) = \psi_j(u, 0)$, this proves the lemma.

Lemma 2

$$\frac{\partial x^j}{\partial u^\alpha}(u(\lambda x)) = \lambda \frac{\partial x^j}{\partial u^\alpha}(u(x)), \quad \frac{\partial u^\alpha}{\partial x^j}(\lambda x) = \frac{1}{\lambda} \frac{\partial u^\alpha}{\partial x^j}(x).$$

Proof. It follows from Lemma 1 that

$$\frac{\frac{\partial x^j}{\partial u^\alpha}(u^1 + \mu, \dots, u^n + \mu)}{x^j(u^1 + \mu, \dots, u^n + \mu)} = \frac{\frac{\partial x^j}{\partial u^\alpha}(u^1, \dots, u^n)}{x^j(u^1, \dots, u^n)}.$$

Therefore we have

$$\begin{aligned} \frac{\partial x^j}{\partial u^\alpha}(u(\lambda x)) &= \frac{\partial x^j}{\partial u^\alpha}(u^1(x) + \mu, \dots, u^n(x) + \mu) = \\ &= \frac{\partial x^j}{\partial u^\alpha}(u(x)) \frac{\lambda x^j(u(x))}{x^j(u(x))} = \lambda \frac{\partial x^j}{\partial u^\alpha}(u(x)), \quad \lambda = e^{-r\mu}, \end{aligned}$$

which proves the first assertion of the lemma. Since $\frac{\partial u^\alpha}{\partial x^j} \frac{\partial x^j}{\partial u^\beta} = \delta_\beta^\alpha$, the second assertion follows from the first one. This proves Lemma 2 and finishes the proof of Theorem 1.

Given a quasihomogeneous solution to the associativity equations (1) with a constant invertible matrix $(\eta^{\alpha\beta})$, one may expand it to a non-semisimple Frobenius manifold as it is explained in §5.

4 Examples

We present a couple of examples. The first of them is the simplest solution from an infinite family provided by Theorem 1 and the second example demonstrates that there are many other solutions with such spectral curves and which are non given by Theorem 1.

EXAMPLE 1. Let Γ is formed by two spheres Γ_1 and Γ_2 which intersect at a pair of points (see Fig. 1):

$$\{a, -a \in \Gamma_1\} \sim \{a, -a \in \Gamma_2\}.$$

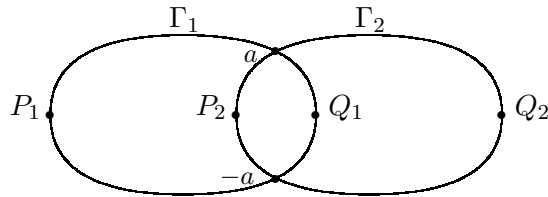


Fig. 1

The arithmetic genus of Γ equals one: $g_a(\Gamma) = 1$.

We consider the case when $n = 2$ and $l = 1$, i.e. the Baker–Akhiezer function is normalized at one point r . We put $r \in \Gamma_2$ and $\psi_2(r) = 1$.

The function ψ takes the form

$$\psi_1 = e^{u^1 z_1} f_0(u^1, u^2), \quad \psi_2 = e^{u^2 z_2} \left(g_0(u^1, u^2) + \frac{g_1(u^1, u^2)}{z_2 - c} \right)$$

and the compatibility conditions read $\psi_1(a) = \psi_2(a), \psi_1(-a) = \psi_2(-a)$. This implies

$$\begin{aligned} \psi_1 &= e^{u^1 z_1} \left(\frac{2a(c-r)e^{au^1+(a-r)u^2}}{(a+c)(a-r)e^{2au^2} - (a+r)(a-c)e^{2au^1}} \right), \\ \psi_2 &= e^{u^2 z_2} \left(\frac{e^{-ru^2}((a-c)e^{2au^1} + (a+c)e^{2au^2})(c-r)}{(a+c)(a-r)e^{2au^2} - (a-c)(a+r)e^{2au^1}} + \right. \\ &\quad \left. \frac{1}{z_2 - c} \frac{(a^2 - c^2)(r-c)e^{-ru^2}(e^{2au^1} - e^{2au^2})}{(a+c)(r-a)e^{2au^2} + (a-c)(a+r)e^{2au^1}} \right). \end{aligned}$$

The differential Ω is defined by the differentials

$$\Omega_1 = \frac{\beta}{z_1(z_1^2 - a^2)} dz_1, \quad \Omega_2 = \frac{(z_2^2 - c^2)}{z_2(z_2^2 - a^2)(z_2^2 - r^2)} dz_2.$$

The regularity condition for Ω take the form

$$\text{res}_a \Omega_1 = \text{res}_{-a} \Omega_1 = \frac{\beta}{2a^2} = -\text{res}_a \Omega_2 = -\text{res}_{-a} \Omega_2 = -\frac{(a^2 - c^2)}{2a^2(a^2 - r^2)},$$

and implies

$$\beta = \frac{c^2 - a^2}{a^2 - r^2}. \quad (7)$$

To achieve the Euclidean metric $\eta_{\alpha\beta} = \delta_{\alpha\beta}$ we assume that $\varepsilon_1^2 = \varepsilon_2^2$ which is written as

$$\text{res}_{Q_1} \Omega_1 = -\frac{\beta}{a^2} = \text{res}_{Q_2} \Omega_2 = -\frac{c^2}{r^2 a^2}$$

from which we derive that

$$\beta = \frac{c^2}{r^2}, \quad r = \frac{a}{\sqrt{2 - \frac{a^2}{c^2}}}. \quad (8)$$

By (7) and (8), we have the formula which restores r from free parameters a and c :

$$r = \frac{a}{\sqrt{2 - \frac{a^2}{c^2}}}.$$

To obtain real-valued functions x^1, \dots, x^n we have to assume that $\tau^*(\Omega) = \bar{\Omega}$ for $\tau : z_i \rightarrow \bar{z}_i, i = 1, 2$. This takes place when

$$a^2, c^2, r^2 \in \mathbb{R}.$$

The prepotential takes the form

$$\begin{aligned} F_{a,c}(x^1, x^2) &= \frac{1}{4ac} \left(2x_2 \sqrt{(a^2 - c^2)x_1^2 + c^2x_2^2} \right. \\ &\quad \left. + 2cx_1^2 \log \left(-\frac{cx_2 + \sqrt{(a^2 - c^2)x_1^2 + c^2x_2^2}}{x_1} \right) - \sqrt{2c^2 - a^2}(x_1^2 + x_2^2) \right. \\ &\quad \left. \times \log \left(c^2(x_1^2 - 3x_2^2) + a^2(x_2^2 - x_1^2) - 2x_2 \sqrt{2c^2 - a^2} \sqrt{(a^2 - c^2)x_1^2 + c^2x_2^2} \right) \right) \end{aligned}$$

and satisfies the associativity equations with $\eta_{\alpha\beta} = \delta_{\alpha\beta}$.

For $a = 1, c = \frac{2}{\sqrt{7}}$ the formulas for coordinates and correlators are rather simple:

$$\begin{aligned} x^1 &= \frac{4(7 - \sqrt{7})e^{u^1 - u^2}}{(21 - 6\sqrt{7})e^{2u^1} + (7 + 2\sqrt{7})e^{2u^2}}, \\ x^2 &= \frac{e^{-2u^2}(3(\sqrt{7} - 3)e^{2u^1} + (5 + \sqrt{7})e^{2u^2})}{3(\sqrt{7} - 2)e^{2u^1} + (2 + \sqrt{7})e^{2u^2}}, \\ c_{111} &= -\frac{9x_1^8 + 51x_1^6x_2^2 + 88x_1^4x_2^4 + (2x_1^2x_2^3 + 4x_2^5)\sqrt{(3x_1^2 + 4x_2^2)^3} + 48x_1^2x_2^6}{2x_1(3x_1^4 + 7x_1^2x_2^2 + 4x_2^4)^2}, \\ c_{112} &= \frac{9x_1^6x_2 + 15x_1^4x_2^3 - 8x_1^2x_2^5 + (2x_1^2x_2^2 + 4x_2^4)\sqrt{(3x_1^2 + 4x_2^2)^3} - 16x_2^7}{2(3x_1^4 + 7x_1^2x_2^2 + 4x_2^4)^2}, \\ c_{122} &= -\frac{9x_1^7 + 15x_1^5x_2^2 - 8x_1^3x_2^4 + (2x_1^3x_2 + 4x_1x_2^3)\sqrt{(3x_1^2 + 4x_2^2)^3} - 16x_1x_2^6}{2(3x_1^4 + 7x_1^2x_2^2 + 4x_2^4)^2}, \\ c_{222} &= \frac{-27x_1^6x_2 - 16x_2^7 - 72x_1^2x_2^5 + (4x_1^2x_2^2 + 2x_1^4)\sqrt{(3x_1^2 + 4x_2^2)^3} - 81x_1^4x_2^3}{2(3x_1^4 + 7x_1^2x_2^2 + 4x_2^4)^2}. \end{aligned}$$

EXAMPLE 2. Let Γ be the same as in Example 1. In difference with Example 1 we assume that

$$P_1 = \infty \in \Gamma_1, \quad P_2 = 0 \in \Gamma_1, \quad Q_1 = \infty \in \Gamma_2, \quad Q_2 = 0 \in \Gamma_2,$$

the normalization point $R = r$ lies in Γ_1 and the divisor of poles $D = c$ lie in Γ_2 (see Fig. 2). Therewith we do not assume that the intersection points have the same coordinates:

$$a \sim b, \quad -a \sim -b, \quad \pm a \in \Gamma_1, \quad \pm b \in \Gamma_2, \quad a \neq b.$$

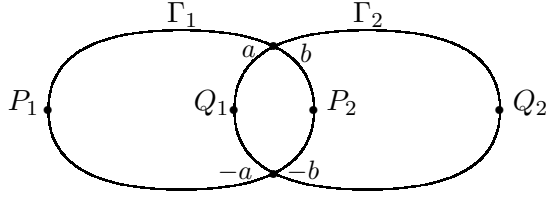


Fig. 2

We take the Baker–Akhiezer function in the form

$$\psi_1 = e^{u^1 2z_1 + \frac{u^2}{2z_1}} f(u), \quad \psi_2 = g_0(u) + \frac{g_1(u)}{z_2 - c}.$$

The differential Ω is defined by the differentials

$$\Omega_1 = \frac{z_1}{(z_1^2 - a^2)(z_1^2 - r^2)} dz_1, \quad \Omega_2 = \frac{(z_2^2 - c^2)}{z_2(z_2^2 - b^2)} dz_2.$$

We have the regularity condition:

$$\text{res}_a \Omega_1 = \text{res}_{-a} \Omega_1 = \frac{1}{2(a^2 - r^2)} = -\text{res}_b \Omega_2 = -\text{res}_{-b} \Omega_2 = -\frac{(b^2 - c^2)}{2b^2},$$

and the Euclidean condition: $\varepsilon_1^2 = \varepsilon_2^2$:

$$\text{res}_{Q_1} \Omega_2 = -1 = \text{res}_{Q_2} \Omega_2 = \frac{c^2}{b^2}.$$

These conditions are satisfied if and only if $b = \pm ic$ and $a^2 - r^2 = -\frac{1}{2}$. We put

$$b = i, \quad c = -1, \quad a = \frac{i}{2}, \quad r = \frac{1}{2}$$

and obtain

$$x^1 = e^{-u^1 - u^2} (\cos(u^1 - u^2) + \sin(u^1 - u^2)),$$

$$x^2 = e^{-u^1 - u^2} (\cos(u^1 - u^2) - \sin(u^1 - u^2)).$$

This gives us the Darboux–Egoroff metric

$$ds^2 = (dx^1)^2 + (dx^2)^2 = 4e^{-2(u^1 + u^2)} \left((du^1)^2 + (du^2)^2 \right)$$

and a quasihomogeneous solution to the associativity equations (1) because as in the case of Theorem 1 we have

$$x^i(u^1 + \mu, u^2 + \mu) = e^{-2\mu} x^i(u^1, u^2), \quad i = 1, 2.$$

Indeed, this solution is very simple and the prepotential $F(x^1, x^2)$ equals

$$F(x^1, x^2) = -\frac{1}{8} \left((x^1)^2 + (x^2)^2 \right) \log \left((x^1)^2 + (x^2)^2 \right).$$

Moreover it is included in a linear pencil of quasihomogeneous functions

$$F_q(x^1, x^2) = q \left((x^1)^2 + (x^2)^2 \right) \arctan \left(\frac{x^1}{x^2} \right) - \frac{1}{8} \left((x^1)^2 + (x^2)^2 \right) \log \left((x^1)^2 + (x^2)^2 \right), \quad q \in \mathbb{R},$$

which satisfy the associativity equations with $\eta_{\alpha\beta} = \delta_{\alpha\beta}$.

The correlators for F are very simple:

$$c_{111} = -\frac{3}{2} \frac{x^1}{(x^1)^2 + (x^2)^2} + \frac{(x^1)^3}{\left((x^1)^2 + (x^2)^2 \right)^2},$$

$$c_{112} = -\frac{1}{2} \frac{x^2}{(x^1)^2 + (x^2)^2} + \frac{(x^1)^2 x^2}{\left((x^1)^2 + (x^2)^2 \right)^2},$$

and the formulas for c_{122} and c_{222} are obtained from the previous ones by permutation of indices $1 \leftrightarrow 2$.

5 An algebraic lemma

Lemma 3 *Let $F(t^1, \dots, t^n)$ be a solution to the associativity equations with the constant metric $\eta_{\alpha\beta}$. Then the function*

$$\tilde{F}(t^0, t^1, \dots, t^n, t^{n+1}) = \frac{1}{2} \left(\eta_{\alpha\beta} t^\alpha t^\beta t^0 + (t^0)^2 t^{n+1} \right) + F(t^1, \dots, t^n)$$

satisfies the associativity equations (1) with the metric $\tilde{\eta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \eta & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and the associative algebra generated by $e_0, e_1, \dots, e_n, e_{n+1}$ with the multiplication law

$$e_i \cdot e_j = c_{ij}^k e_k, \quad c_{ij}^k = \tilde{\eta}^{kl} \frac{\partial^3 \tilde{F}}{\partial t^l \partial t^i \partial t^j},$$

has the unity e_0 :

$$e_0 \cdot e_k = e_k \quad \text{for all } k = 0, \dots, n+1,$$

and the nilpotent element e_{n+1} :

$$e_{n+1}^2 = 0.$$

Moreover if F is quasihomogeneous and $d_\alpha + d_\beta = c$ for all α, β such that $\eta_{\alpha\beta} \neq 0$ then \tilde{F} is also quasihomogeneous with $d_0 = d_F - c, d_{n+1} = 2c - d_F$ and the same values of $d_\alpha, \alpha = 1, \dots, n$, as for F .

The proof of this lemma is straightforward.

Applying this procedure to the examples from §4, we obtain four-dimensional Frobenius manifolds M with coordinates $t^0, t^2 = x^1, t^3 = x^2, t^3$. The element e_0 serves as the unity and the element e_3 is nilpotent in any tangent algebra $T_t M$: $e_{n+1}^2 = 0$. In these examples we have $d_F = 2, d_1 = d_2 = 1$ and therefore $d_0 = 0$ and $d_3 = 2$.

These examples give two-dimensional deformations of the cohomology ring of $\mathbb{C}P^2 \sharp \mathbb{C}P^2$. Indeed we have the standard generators e_0, \dots, e_3 in $H^*(\mathbb{C}P^2 \sharp \mathbb{C}P^2; \mathbb{C})$: $e_0 \in H^0, e_1, e_2 \in H^2, e_3 \in H^4, e_1^2 = e_2^2 = e_3, e_1 e_2 = 0$. We also have the identity $d_i = \frac{\deg e_i}{2}$. These deformations change the multiplication rules for two-dimensional classes by adding two-dimensional terms: $e_i e_j = e_3 + c_{ij}^k(t) e_k, i, j = 1, 2$.

We remark that in the Seiberg–Witten theory the associativity equations also appear even in a more general setting: the matrix η is not necessarily constant and the quasihomogeneity condition is lifted [6].

References

- [1] Barannikov, S., and Kontsevich, M.: Frobenius manifolds and formality of Lie algebras of polyvector fields. *Internat. Math. Res. Notices* (1998), no. 4, 201–215.

- [2] Dijkgraaf, R., Verlinde, E., and Verlinde, H.: Notes on topological string theory and 2D gravity. Nucl. Phys. B **352** (1991), 59–86.
- [3] Dubrovin, B.: Integrable systems in topological field theory. Nucl. Phys. B **379** (1992), 627–689.
- [4] Dubrovin, B.: Geometry of 2D topological field theories. Lecture Notes in Math., **1620**. Springer, Berlin, 1995, 120–348.
- [5] Krichever, I.M.: Algebraic-geometric n -orthogonal curvilinear coordinate systems and the solution of associativity equations. Funct. Anal. Appl. **31**:1 (1997), 25–39.
- [6] Marshakov, A., Mironov, A., and Morozov, A.: WDVV-like equations in $N = 2$ SUSY Yang-Mills theory. Phys. Lett. B **389** (1996), 43–52.
- [7] Mironov, A.E., and Taimanov, I.A.: Orthogonal curvilinear coordinate systems corresponding to singular spectral curves. Proc. of the Steklov Institute of Math. (2006).
- [8] Shramchenko, V.: "Real doubles" of Hurwitz Frobenius manifolds. Comm. Math. Phys. **256** (2005), 635–680.
- [9] Witten, E.: On the structure of the topological phase of two-dimensional gravity. Nucl. Phys. B **340** (1990), 281–332.
- [10] Zakharov, V.E.: Description of the n -orthogonal curvilinear coordinate systems and Hamiltonian integrable systems of hydrodynamic type, I: Integration of the Lamé equation. Duke Math. J. **94** (1998), 103–139.